Cell decomposition and definable functions for weak p-adic structures

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Abstract

We develop a notion of cell decomposition suitable for studying weak p-adic structures (reducts of p-adic fields where addition and multiplication are not (everywhere) definable).

1 Introduction and first definitions

Results for real fields have always been a big source of inspiration for the study of p-adic fields. An example of this is the concept of o-minimality, see e.g van den Dries [15], which inspired Haskell and Macpherson [5] to develop a similar concept, P-minimality, for p-adic fields. A difference between those concepts is that o-minimality also covers reducts of real closed fields, see Peterzil [10, 13, 12], while P-minimality focuses on expansions of the language of valued fields. To fill this gap in the study of p-adic fields, we need to consider reducts $(K, \mathcal{L})$ of $(K, \mathcal{L}_{\text{ring}})$, where K is a p-adically closed field, and the $\mathcal{L}$-definable subsets of K are exactly the $\mathcal{L}_{\text{ring}}$-definable (semi-algebraic) subsets of K.

A first step towards understanding such structures is to describe the boundaries of our 'playing field': identify the relations and functions that, as a bare minimum, would have to be definable in such a structure. In our paper [2], we concluded that any reduct of $(K, \mathcal{L}_{\text{ring}})$ where the relations

$$R_{n,m}(x, y, z) \leftrightarrow y - x \in zQ_{n,m}$$

are definable, would fit inside this framework. The sets $Q_{n,m}$, which for $K = \mathbb{Q}_p$ can be defined as $\cup_{k\in\mathbb{Z}} p^{kn}(1 + p^m\mathbb{Z}_p)$, are a variation on the sets of n-th powers P_n that one encounters in the study of p-adic semi-algebraic sets. A more general definition will be given in section 1.1.

Now that we know which structures we want to consider, the second step will be to describe their definable sets. Historically, cell decomposition has proved to be a very useful tool in studying definability questions. Examples include o-minimal cell decomposition in the real case, and Denef's cell decomposition for p-adic semi-algebraic sets, which can be stated as follows:

Theorem 1.1 (Denef, [3, 4]) *Let K be a finite field extension of $\mathbb{Q}_p$. Any semi-algebraic set $X \subseteq K^{k+1}$ can be partitioned as a finite union of cells of the form*

$$\{(x,t) \in D \times K \mid \operatorname{ord} a_1(x) \;\square_1\; \operatorname{ord}(t - c(x)) \;\square_2\; \operatorname{ord} a_2(x), t - c(x) \in \lambda P_n\},$$

where D is a semi-algebraic subset of K^k and $c(x), a_i(x)$ are semi-algebraic functions.

Among other applications, Denef used this result to give a new proof of Macintyre's quantifier elimination result [9]. The result was also important for Mourgues [11] result on P-minimal cell decomposition.

In [2], we showed that the relation $D^{(3)}(x,y,z) \leftrightarrow \operatorname{ord}(x-y) < \operatorname{ord}(z-y)$ is definable in the language $\mathcal{L}_M := (\{R_{n,m}\}_{n,m})$. Because of this, cells $C \subset K$ are $\mathcal{L}_M$-definable, which is what we wanted. However, to get a language that is more convenient to work with, we will replace $D^{(3)}$ by the slightly stronger relation $D^{(4)}$, defined as

$$D^{(4)}(x,y,z,t) \leftrightarrow \operatorname{ord}(x-y) < \operatorname{ord}(z-t).$$

The resulting language $\mathcal{L}_{\text{dist}} = (D^{(4)}, \{R_{n,m}\}_{n,m})$ is strictly stronger than $\mathcal{L}_M$, as there are sets which are $\mathcal{L}_{\text{dist}}$-definable, but not $\mathcal{L}_M$-definable [8].

Our definition of cells is inspired by Denef's p-adic cells, but with some modifications, the first being that we use the sets $Q_{n,m}$ instead of the usual P_n. A second difference is that we will only require the relation $\operatorname{ord} a_i(x) < \operatorname{ord} t$ to be definable, and not necessarily the function $a_i(x)$ itself. This change is motivated by the observation that the function $(x,y) \mapsto x - y$ is not necessarily definable in all languages that contain a symbol for the relation $D^{(4)}$. We will call this *weak* cell decomposition as opposed to (strong) cell decomposition results that only use definable functions.

While working on the cell decomposition results presented in this paper, we noticed that many of those results were valid for a much wider class of fields than just p-adically closed fields. In particular, we do not need to assume that the field is henselian, and our results will work in any characteristic. For this reason, even though our original motivation was the study of p-adically closed fields, we present our results for $(\mathbb{F}_q, \mathbb{Z})$-fields: valued fields with residue field isomorphic to $\mathbb{F}_q$ and value group elementarily equivalent to $\mathbb{Z}$. So this paper is really about cell decomposition techniques for expansions of $(K, \mathcal{L}_{\text{dist}})$, where K is an $(\mathbb{F}_q, \mathbb{Z})$-field.

Let us now give a brief overview of the contents of this paper. We first explain the ideas behind the sets $Q_{n,m}$ in section 1.1, and then give a formal definition of our concept of cells in section 1.2. In section 2, we state some cell decomposition results valid for all expansions of $\mathcal{L}_{\text{dist}}$, and we show how these results can be used to study the language $\mathcal{L}_{\text{dist}}$ itself.

In our definition of cells, we made a distinction between weak and strong cell decomposition, depending on whether or not the functions used were definable. In section 3 we investigate the definable functions of structures that admit weak cell decomposition. In particular, we will focus on the existence of definable Skolem functions. The reason for this is a result by Mourgues for P-minimal fields, stating that a structure has (strong) cell decomposition if and only if it admits definable Skolem functions. When restricting our attention to p-adically closed fields, and under the assumption that multiplication by a sufficient number of scalars is definable, we can obtain a similar result for expansions of $\mathcal{L}_{\text{dist}}$.

However, this result is not as strong as it seems to be. In P-minimality, requiring the existence of definable Skolem functions is a relatively minor assumption, as there are no known examples of structures that do not have such functions. For the weaker structures we study, we get a different picture: $(K, \mathcal{L}_{\text{dist}})$ itself is an example of a structure having no definable Skolem functions. So the really interesting questions are whether or not every expansion of $(K, \mathcal{L}_{\text{dist}})$ admits weak cell decomposition, and under which conditions a structures would have definable skolem functions.

At this time, we cannot answer the first question, and we can only give a conjecture for the second question. In o-minimality, it is known that any structure where addition is definable, has definable skolem functions. Based on the structures we have studied, our guess would be that an expansion of $\mathcal{L}_{\text{dist}}$ probably has Skolem functions if addition and sufficient scalar multiplication is definable. While we cannot give a proof for this, we have strong indications that structures where these requirements are not satisfied can never have definable Skolem functions. Section 4 provides an example of a structure where we have full scalar multiplication, and where addition is definable on large open sets, but which still does not admit definable Skolem functions.

The expansions considered in this paper are still very basic. Adding a symbol for either addition [7] or (restricted) multiplication [8] is also possible, but for this we refer to [7] and [8].

1.1 Definition of the sets $Q_{n,m}$

The field of p-adic numbers admits elimination of quantifiers in the language $\mathcal{L}_{\text{Mac}}$, which is the ring language, extended with predicates P_n for the sets of n-th powers. Note that by Hensel's lemma, any $x \in \mathbb{Q}_p$ can be written, using a sufficiently large $r \in \mathbb{N}$, as $x = \lambda \cdot u$, with $u \in (1 + p^r\mathbb{Z}_p) \cap P_n$, so that cosets λP_n encode certain information concerning the value group and the angular components of their elements. This is essentially the reason why this language admits elimination of quantifiers.

However, this only works because $\mathbb{Q}_p$ and other p-adically closed fields are Henselian. In general, additive-multiplicative congruences (amc-structures), as proposed by Basarab and Kuhlmann [1, 6] are necessary to obtain quantifier elimination in a definitional expansion of the valued field language. Amc-structures are quotient groups $K^\times/(1 + I)$, for ideals $I \subset \mathcal{O}_K$ (where $\mathcal{O}_K$ is

the valuation ring of K). For example, if $I = \mathcal{M}_K$, the maximal ideal of $\mathcal{O}_K$, then the corresponding amc-structure encodes information about the value group and the angular component modulo π.

We propose to use a variation on amc-structures, that encodes similar information as the sets P_n, even when Hensel's lemma does not hold. In particular, we will consider the following class of fields.

Definition 1.2 *Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{Z}$ the ordered abelian group of integers. We define a $(\mathbb{F}_q, \mathbb{Z})$-field to be a valued field with residue field isomorphic to $\mathbb{F}_q$ and value group elementarily equivalent to $\mathbb{Z}$.*

Fix a $(\mathbb{F}_q, \mathbb{Z})$-field K, fix an element π_K with smallest positive order, and write $\mathcal{M}_K$ for the maximal ideal of the valuation ring $\mathcal{O}_K$ of K. For each integer $n > 0$, let P_n be the set of nonzero n-th powers in K. For each $m > 0$, we will define sets $Q_{n,m}$ using angular component maps. The following lemma shows that such maps exist and that they can be defined in a unique way.

Lemma 1.3 *For each integer $m > 0$, there is a unique group homomorphism*

$$ac_m : K^\times \to (\mathcal{O}_K \bmod \pi_K^m)^\times$$

such that $ac_m(\pi_K) = 1$ and such that $ac_m(u) \equiv u \bmod (\pi_K)^m$ for any unit $u \in \mathcal{O}_K$.

Proof. Put $N_m := (q-1)q^{m-1}$ and let U be the set $P_{N_m} \cdot \mathcal{O}_K^\times$. Note that $K^\times$ equals the finite disjoint union of the sets $\pi_K^\ell \cdot U$ for integers ℓ with $0 \leqslant \ell \leqslant N_m - 1$. Hence, any element y of $K^\times$ can be written as a product of the form $\pi_K^\ell x^{N_m} u$, with $u \in \mathcal{O}_K^\times$, $\ell \in \{0, \ldots, N_m - 1\}$, and $x \in K^\times$.

Since ac_m is required to be a group homomorphism to a finite group with N_m elements, it must send P_{N_m} to 1. Also note that the projection $\mathcal{O}_K \to \mathcal{O}_K \bmod \pi_K^m$ (which is a ring homomorphism), induces a natural group homomorphism $p : \mathcal{O}_K^\times \to (\mathcal{O}_K \bmod \pi_K^m)^\times$. Now if we write $y = \pi_K^\ell x^{N_m} u$, we see that ac_m must satisfy

$$\mathrm{ac}_m(y) = p(u), \tag{1}$$

which implies that the map ac_m is uniquely determined if it exists. Moreover, we claim that we can use (1) to define ac_m. This is certainly a well defined group homomorphism: if one writes $y = \pi_K^\ell \tilde{x}^{N_m} \tilde{u}$ for some other $\tilde{u} \in \mathcal{O}_K^\times$ and $\tilde{x} \in K^\times$, then clearly $p(u) = p(\tilde{u})$. It is also clear that this homomorphism sends π_K to 1 and satisfies our requirement that $\mathrm{ac}_m(u) \equiv u \bmod (\pi_K)^m$ for any unit $u \in \mathcal{O}_K$. $\square$

Using these angular component maps, we can define sets $Q_{n,m}$, for any $m, n > 0$, as follows:

$$Q_{n,m} := \{x \in P_n \cdot (1 + \mathcal{M}_K^m) \mid \mathrm{ac}_m(x) = 1\}.$$

Note that $Q_{n,m}$ is an open subgroup of finite index of $K^\times$ (for the valuation topology), and that $Q_{n,m}$ is definable in the language of valued fields $(+, -, \cdot, |)$

by the above construction of ac_m. For example if $K = \mathbb{Q}_p$, then $Q_{n,m}$ is just the set $\bigcup_{k\in\mathbb{Z}} p^{kn}(1 + p^m\mathbb{Z}_p)$.

For any element $\lambda \in K$, let $\lambda Q_{n,m}$ denote the set $\{\lambda t \mid t \in Q_{n,m}\}$. We will sometimes use the alternative notation $\rho_{n,m}(x) = \lambda$ to express that $x \in \lambda Q_{n,m}$. The relation between $\rho_{n,m}(x + y)$, $\rho_{n,m}(x)$ and $\rho_{n,m}(y)$ is investigated in the lemma below (the proof is left to the reader).

Lemma 1.4 *Put $\delta \in \{-1, 1\}$. Suppose that $\rho_{n,m}(a) = \lambda$ and $\rho_{n,m}(b) = \mu$, then $\rho_{n,m}(a + \delta b)$ equals*

$$\begin{cases} \lambda & \text{if} \quad m + \operatorname{ord} a \leqslant \operatorname{ord} b, \\ \rho_{n,m}(\lambda + \delta\mu\pi^{rn}), \text{ with } rn = \operatorname{ord}\left(\frac{\mu a}{\lambda b}\right) & \text{if} \quad -m + \operatorname{ord} b < \operatorname{ord} a < \operatorname{ord} b, \\ \rho_{n,m}(\lambda + \delta\mu) & \text{if} \quad \operatorname{ord} a = \operatorname{ord} b = \operatorname{ord}(a + \delta b). \end{cases}$$

1.2 Definition of weak p-adic cells

Let K be an $(\mathbb{F}_q, \mathbb{Z})$-field. As stated in the introduction, we will be working with functions $f : K^k \to K$ for which the relation

$$\operatorname{ord} f(x) < \operatorname{ord} t$$

is definable for all $(x, t) \in K^{k+1}$. We call these *order-definable* functions. Note that f is not required to be a definable function. However, the following relations are always definable if the relation $\operatorname{ord} x < \operatorname{ord} y$ is definable in our language:

Lemma 1.5 *Let $\mathcal{L}$ be a language where the relation $D^{(2)}(x, y) := \operatorname{ord} x < \operatorname{ord} y$ is definable. Let $a_1(x), a_2(x)$ be two functions $K^k \to K$ that are order-definable in $\mathcal{L}$. The following relations are definable:*

(1) $\operatorname{ord} x < \operatorname{ord} y + k$*, for any* $k \in \mathbb{Z}$*;*

(2) $\operatorname{ord} a_1(x) \square \operatorname{ord} t$*;*

(3) $\operatorname{ord} a_1(x) \square \operatorname{ord} a_2(x) + k$*, for any* $k \in \mathbb{Z}$*;*

where $\square$ may denote $<, \leqslant, =, \geqslant$ or $>$.

Proof. (1) For $k < 0$, the relation $\operatorname{ord} x + k > \operatorname{ord} y$ is equivalent with

$$(\exists u_1) \dots (\exists u_{-k})[\operatorname{ord} x > \operatorname{ord} u_1 > \operatorname{ord} u_2 > \dots > \operatorname{ord} u_{-k} > \operatorname{ord} y].$$

For $k > 0$, it is equivalent with

$$(\forall u_1) \dots (\forall u_k) \left[(\operatorname{ord} x < \operatorname{ord} u_1 < \dots < \operatorname{ord} u_k) \to \operatorname{ord} u_k > \operatorname{ord} y\right].$$

(2) For example if $\square$ denotes '=', we can define the relation $\operatorname{ord} a_1(x) = \operatorname{ord} t$ as

$$\neg(\operatorname{ord} a_1(x) < \operatorname{ord} t) \wedge (\forall u)(\operatorname{ord} t < \operatorname{ord} u \to \operatorname{ord} a_1(x) < \operatorname{ord} u).$$

The other cases can be derived from this.
(3) The relation $\operatorname{ord} a_1(x) < \operatorname{ord} a_2(x)$ is equivalent with

$$(\exists t)(\operatorname{ord} a_1(x) < \operatorname{ord} t \wedge \operatorname{ord} a_2(x) = \operatorname{ord} t).$$

The rest can be derived from (1) and (2). $\square$
For our notion of cell decomposition, a p-adic cell will be a set of the following form.

Definition 1.6 (p-adic cell) *Let $\mathcal{L}$ be an expansion of $(D^{(2)}, \{R_{n,m}\}_{n,m})$. For each $k > 0$, let Δ_k be a collection of (not necessarily $\mathcal{L}$-definable) functions $K^k \to K$, and put $\Delta = \cup_k \Delta_k$.*
*An $(\mathcal{L}, \Delta)$-*precell *in K^k is a set $\{x \in K^k \mid \phi(x)\}$, where $\phi(x)$ is a boolean combination of relations of the forms*

$$\operatorname{ord} a_1(x) \;<\; \operatorname{ord} a_2(x) + \ell, \quad \textit{and} \quad b_1(x) - b_2(x) \in \lambda Q_{n,m},$$

where the $a_i(x)$ are functions in Δ_k, $\ell \in \mathbb{Z}$ and the $b_i(x)$ are quantifier free definable functions for $\mathcal{L}$.
A (p-adic) $(\mathcal{L}, \Delta)$-definable cell $C_c^D(a_1, a_2, \lambda) \subseteq K^{k+1}$ *is a set of the following form:*

$$\{(x,t) \in D \times K \mid \operatorname{ord} a_1(x) \;\square_1\; \operatorname{ord} t - c(x) \;\square_2\; \operatorname{ord} a_2(x), t - c(x) \in \lambda Q_{n,m}\},$$

where $\lambda \in K$, D is an $(\mathcal{L}, \Delta)$-precell in K^k, $\square_i$ denotes '$<$' or 'no condition', and the $a_i(x)$ are functions from Δ_k. We call the function $c(x)$ the center of the cell and we require $c(x)$ to be a quantifier free definable function for $\mathcal{L}$.

Definition 1.7 *Let $\mathcal{L}$ be a language expanding $(D^{(2)}, \{R_{n,m}\}_{n,m})$, and K an $(\mathbb{F}_q, \mathbb{Z})$-field, which we consider as an $\mathcal{L}$-structure.*
We say that $\mathcal{L}$ allows cell decomposition (with definable centers) *in K if there exists a set of functions Δ such that for every $k \in \mathbb{N}$, any $\mathcal{L}$-definable subset of K^{k+1} can be partitioned into ($\mathcal{L}$-definable) p-adic $(\mathcal{L}, \Delta)$-cells. More specifically,*

- *The decomposition is* strong *if every $f \in \Delta$ is $\mathcal{L}$-definable.*
- *The decomposition is* weak *if every $f \in \Delta$ is order-definable for $\mathcal{L}$.*

The languages we study in this paper only allow weak cell decomposition. Cell decomposition for semi-algebraic sets and semilinear sets are (after a few straightforward adaptations) examples of strong cell decomposition.

2 Cell Decomposition results

Let $\mathcal{L}$ be a language expanding the language $\mathcal{L}_{\text{dist}}$ (which we defined in the introduction). Every boolean combination of weakly $\mathcal{L}$-definable cells can be partitioned into a finite number of weakly $\mathcal{L}$-definable cells. This is an easy consequence of the following theorem.

Theorem 2.1 *Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{dist}$. Let Δ be a collection of functions that are order-definable in $\mathcal{L}$. Let A_1, A_2 be $(\mathcal{L}, \Delta)$-definable cells with centers c_1, resp. c_2. The intersection $A_1 \cap A_2$ can be written as a finite union of disjoint weak $\mathcal{L}$-cells A each of which has a center which is a restriction of either c_1 or c_2.*

Proof. By partitioning C_1 and C_2 further if necessary, we may suppose that they both use $Q_{n,m}$ with the same positive integers m, n, that is, that C_i is of the form

$$\{(x,t) \in D_i \times K \mid \operatorname{ord} a_{1i}(x) \ \square_{1i} \ \operatorname{ord}(t - c_i) \ \square_{2i} \ \operatorname{ord} a_{2i}(x), \ t - c_i \in \lambda_i Q_{n,m}\}$$

for $i \in \{1, 2\}$, where the symbols have their meaning as in Definition 1.6. Using Lemma 1.5, we can find a finite partitioning of C_1 in cells with the same center, such that on such a cell one of the following conditions holds for $k = 1 + m + n + \sum_{i,j=1,2} |k_{ij}|$ and some integer ℓ_1 with $-k \leq \ell_1 \leq k$.

$$\begin{array}{rclr} \operatorname{ord}(t - c_1) & > & \operatorname{ord}(c_2 - c_1) + k, & (\mathrm{I})_k \\ \operatorname{ord}(t - c_1) & < & \operatorname{ord}(c_2 - c_1) - k, & (\mathrm{II})_k \\ \operatorname{ord}(t - c_1) + \ell_1 & = & \operatorname{ord}(c_2 - c_1). & (\mathrm{III})_{\ell_1} \end{array}$$

Hence, we may suppose that one of these conditions holds for C_1. Note that $(\mathrm{I})_k$ and $(\mathrm{II})_k$ imply respectively

$$\begin{array}{rclclr} \operatorname{ord}(t - c_1) & > & \operatorname{ord}(c_2 - c_1) + k & = & \operatorname{ord}(t - c_2) + k, & (\mathrm{i})_k \\ \operatorname{ord}(t - c_2) & = & \operatorname{ord}(t - c_1) & < & \operatorname{ord}(c_1 - c_2) - k. & (\mathrm{ii})_k \end{array}$$

If $(\mathrm{I})_k$ holds on C_1, put

$$W = \{x \in D_2 \mid \operatorname{ord} a_{12}(x) \square_{12} \operatorname{ord}(c_1 - c_2) \square_{22} \operatorname{ord} a_{22}(x)\}.$$

Then one has, if $C_1 \cap C_2$ is nonempty, that

$$C_1 \cap C_2 = (W \times K) \cap C_1,$$

which can easily be seen to be a finite disjoint union of $(\mathcal{L}, \Delta)$-cells of the desired form. If $(\mathrm{II})_k$ holds on C_1, then we may suppose, up to partitioning C_2, that $(\mathrm{II})_k$ holds for all $(x,t) \in C_1$ and all $(x,t) \in C_2$. Also, we find that $\rho_{n,m}(t - c_1) = \rho_{n,m}(t - c_2)$, so either $C_1 \cap C_2$ is empty or $C_1 \cap C_2$ consists of all points $(x,t) \in (D_1 \cap D_2) \times K$ satisfying the conditions

$$\max_{i \in I}\{\operatorname{ord} a_{1i}(x)\} < \operatorname{ord}(t - c_1) < \min_{i=1,2}\{\operatorname{ord} a_{2i}(x)\},$$

and

$$t - c_1 \in \lambda_1 Q_{n,m},$$

where I consists of i such that $\square_{1i}$ is the condition $<$, and where the maximum over the empty set is $-\infty$. We know by Lemma 1.5 that relations of the form

$\operatorname{ord} a_{ij} < \operatorname{ord} a_{lk}$ are definable. It is then easy to see that $C_1 \cap C_2$ can be partitioned into a finite number of disjoint $(\mathcal{L}, \Delta)$-cells, which finishes the proof for this case.

We may suppose by symmetry (that is, up to reversing the role of C_1 and C_2) that, if $(\mathrm{III})_{\ell_1}$ holds on C_1, then also

$$\operatorname{ord}(t - c_1) + \ell_1 \quad = \quad \operatorname{ord}(c_2 - c_1) \quad = \quad \operatorname{ord}(t - c_2) + \ell_2 \quad (\mathrm{iii})_\ell$$

holds with $\ell = (\ell_1, \ell_2)$ and $-k \leq \ell_2 \leq k$. Suppose again that $C_1 \cap C_2$ is nonempty. If one now fixes the residue classes of $c_2 - c_1$ and of $t - c_1$ modulo $Q_{2kn,2kn}$, then the conditions

$$\operatorname{ord}(c_2 - c_1) = \operatorname{ord}(t - c_2) + \ell_2 \text{ and } t - c_2 \in \lambda_2 Q_{n,m}$$

follow automatically from $\operatorname{ord}(t - c_1) + \ell_1 = \operatorname{ord}(c_2 - c_1)$. (The exact relations are described in Lemma 1.4.) Hence, one can easily partition $C_1 \cap C_2$ into finitely many $(\mathcal{L}, \Delta)$-cells. $\square$

One of our main motivations for using cell decomposition is because it is a very useful tool for quantifier elimination. An example is Denef's proof of quantifier elimination for semi-algebraic sets [4]. The following lemma, which is closely inspired by this paper, will be used quite often.

Lemma 2.2 *Let $\mathcal{L}$ be a language expanding $(D^{(2)}, \{R_{n,m}\}_{n,m})$. Let $C_c^D(a_1, a_2, \lambda) \subseteq K^{k+1}$ be a weakly $\mathcal{L}$-definable cell. Suppose that for every $l \in \mathbb{N}$, the set $\{x \in K^k \mid \operatorname{ord} a_1 \equiv l \mod n\}$ can be partitioned as a finite union of precells $\subseteq K^k$. Then the projection*

$$P := \{x \in K^k \mid \exists t : (x, t) \in C_c^D(a_1, a_2, \lambda)\}$$

can be partitioned in a finite number of precells.

Proof. Note that P is in fact equal to the following set:

$$P = \{x \in D \mid \exists \gamma \in \Gamma_K : \operatorname{ord} a_1(x) < \gamma < \operatorname{ord} a_2(x),\ \gamma \equiv \operatorname{ord} \lambda \mod n\}.$$

Thus P is the set of all $x \in D$ satisfying

$$\exists \gamma \in \Gamma_K : \frac{\operatorname{ord}[a_1(x)\lambda^{-1}]}{n} < \gamma < \frac{\operatorname{ord}[a_2(x)\lambda^{-1}]}{n}. \tag{2}$$

Now if $\operatorname{ord} a_1(x)\lambda^{-1} \equiv \zeta \mod n$, for $0 \leqslant \zeta < n$, then condition (2) is equivalent with $\operatorname{ord} a_1(x)\lambda^{-1} + n - \zeta < \operatorname{ord} a_2(x)\lambda^{-1}$, which can be simplified to

$$\operatorname{ord} a_1(x) + n - \zeta < \operatorname{ord} a_2(x). \tag{3}$$

This completes the proof, since D is a precell, (3) is a precell condition and by our assumption, the set $\{x \in K^k \mid \operatorname{ord} a_1(x)\lambda^{-1} \equiv \zeta \mod n\}$ can be partitioned as a finite union of precells. $\square$

2.1 Example: the language $\mathcal{L}_{\text{dist}}$

We will now use the results from the previous section to investigate the language $\mathcal{L}_{\text{dist}}$. More specifically, we will show that the definitional expansion

$$\mathcal{L}'_{\text{dist}} := (\{D_k^{(4)}\}_{k\in\mathbb{Z}}, \{R_{n,m}\}_{n,m}),$$

where

$$D_k^{(4)}(x,y,z,t) \leftrightarrow \text{ord}\,(x-y) < \text{ord}\,(z-t) + k,$$

admits elimination of quantifiers for any $(\mathbb{F}_q,\mathbb{Z})$-field.

Definition 2.3 *We call a polynomial $f(x) \in K[x_1,\ldots,x_k]$ an $\mathcal{L}_{dist}$-polynomial in variables $\{x_1,\ldots x_k\}$ if $f(x)$ has one of the following forms*

$$f(x) = a, \quad \text{or} \quad f(x) = \pi^k(x_i - a), \quad \text{or} \quad f(x) = \pi^k(x_i - x_j),$$

where $a \in K,\ k \in \mathbb{Z}$; $1 \leqslant i,j \leqslant k$.

An $\mathcal{L}_{\text{dist}}$-cell will be a p-adic cell with the following specifications.

Definition 2.4 *Let $\Delta_{dist,k}$ be the set of all $\mathcal{L}_{dist}$-polynomials in k variables, and put $\Delta_{dist} := \cup_{k\geqslant 0}\Delta_{dist,k}$.*
An $\mathcal{L}_{dist}$-cell $\subseteq K^{k+1}$ is an $(\mathcal{L}_{dist},\Delta_{dist})$-definable cell $\subseteq K^{k+1}$ for which the center $c(x)$ is either a constant from K or one of the variables $x_1,\ldots x_k$.

It is an easy consequence of Lemma 1.5 that $\mathcal{L}_{\text{dist}}$- polynomials are order-definable functions. Therefore the following holds for $\mathcal{L}_{\text{dist}}$-cells.

Proposition 2.5 *Let A_1, A_2 be $\mathcal{L}_{dist}$- cells with centers c_1, resp. c_2. The intersection $A_1 \cap A_2$ can be written as a finite union of disjoint $\mathcal{L}_{dist}$-cells A with as center a restriction of either c_1 or c_2.*

Proof. This follows from Theorem 2.1. □

Proposition 2.6 *The language $\mathcal{L}'_{dist}$ admits elimination of quantifiers for any $(\mathbb{F}_q,\mathbb{Z})$-field.*

Proof. It is clear that any $\mathcal{L}_{\text{dist}}$-cell (and any $\mathcal{L}_{\text{dist}}$-precell) is quantifier free definable in $\mathcal{L}'_{\text{dist}}$. Moreover, for any $\mathcal{L}_{\text{dist}}$-polynomial $f(x)$, the relation ord $f(x) \equiv l$ mod n can be partitioned in a finite number of precells. (Indeed, this relation can be written as a finite disjunction of relations $x_i - a \in \lambda Q_{n,m}$ or $x_i - x_j \in \lambda Q_{n,m}$.)

Since the requirements of Lemma 2.2 are satisfied, it is now sufficient to show that any set that is quantifier free definable in $\mathcal{L}'_{\text{dist}}$ can be partitioned as a finite union of $\mathcal{L}_{\text{dist}}$-cells.

By Proposition 2.5, we only need to check that the sets (and complements of these sets)

$$\{x \in K^k \mid D_r^{(4)}(g_1,g_2,g_3,g_4)\} \quad \text{and} \quad \{x \in K^k \mid R_{n,m}(g_1,g_2,g_3)\},$$

with $g_i(x) \in \{x_1, \ldots, x_k\} \cup K$ and $r \in \mathbb{Z}$ can be partitioned as a finite union of $\mathcal{L}_{\text{dist}}$-cells.
The fact that $R_{n,m}(x, y, z)$ is equivalent to

$$\bigvee_{[\lambda] \in \Lambda_{n,m}} R_{n,m}(x, y, \lambda) \wedge R_{n,m}(0, z, \lambda),$$

implies that $\{x \in K^3 \mid R_{n,m}(x, y, z)\}$ can be written as a union of $\mathcal{L}_{\text{dist}}$-cells, by Theorem 2.1. The complement of this set can also be written as a union of disjoint $\mathcal{L}_{\text{dist}}$-cells, since $\{(x, y) \in K^2 \mid \neg R_{n,m}(x, y, \lambda)\}$ can be written as a finite union of (disjoint) sets of the form $\{(x, y) \in K^2 \mid R_{n,m}(x, y, \mu)\}$.

To complete the proof it suffices to check that the set

$$A := \{(x, t) \in K^{k+1} \mid \operatorname{ord}(t - c_1) < \operatorname{ord} \pi^n (t - c_2)\},$$

with $c_1, c_2 \in \{x_1, \ldots, x_k\} \cup K$, can be partitioned as a finite union of cells. We may suppose that $c_1 \neq c_2$. Partition K^{k+1} in the following way:

$$\begin{aligned} K^{k+1} \quad = \quad & \{(x, t) \in K^{k+1} \mid \operatorname{ord}(t - c_1) > \operatorname{ord}(c_1 - c_2)\} \\ & \cup \{(x, t) \in K^{k+1} \mid \operatorname{ord}(t - c_1) < \operatorname{ord}(c_1 - c_2)\} \\ & \cup \{(x, t) \in K^{k+1} \mid \operatorname{ord}(t - c_1) = \operatorname{ord}(c_1 - c_2)\}. \end{aligned} \tag{4}$$

Since $A = A \cap K^{k+1}$, we can write A as a union of sets on which one of the conditions in (4) holds. For example, on

$$B = A \cap \{(x, t) \in K^{k+1} \mid \operatorname{ord}(t - c_1) > \operatorname{ord}(c_1 - c_2)\},$$

we have that $\operatorname{ord}(t - c_2) = \operatorname{ord}(c_1 - c_2)$, and therefore B is equal to the set

$$B = \{(x, t) \in K^{k+1} \mid \operatorname{ord}(c_1 - c_2) < \operatorname{ord}(t - c_1) < \operatorname{ord} \pi^n (c_1 - c_2)\}.$$

It is clear that B can be partitoned as a finite number of $\mathcal{L}_{\text{dist}}$-cells. The other cases are similar. $\square$

Note our strategy: For the given language $\mathcal{L}$, we first try to find a suitable set Δ of order-definable functions, such that we get a system of weak $(\mathcal{L}, \Delta)$-definable cells. To obtain a definitional expansion that has QE, we add symbols to $\mathcal{L}$ such that for each $f \in \Delta$, the relations

$$\operatorname{ord} f(x) < t + k \quad \text{and} \quad \operatorname{ord} f(x) \equiv l \mod n$$

are quantifier free definable in the extended language $\mathcal{L}'$. To obtain QE for $\mathcal{L}'$, it is then sufficient to show that quantifier free $\mathcal{L}'$-definable sets can be partitioned as a finite number of $(\mathcal{L}, \Delta)$-cells. (Unfortunately, this last step may require quite a lot of work.)

3 Cell decomposition and definable (Skolem) functions

3.1 Definable functions

The example of $\mathcal{L}_{\text{dist}}$-definable sets illustrates how we can use cell decomposition to obtain quantifier elimination results for a structure $(K, \mathcal{L})$. Cell decomposition results also provide a lot of information concerning the definable functions of a given structure. For instance, all $\mathcal{L}_{\text{dist}}$-definable functions must have the following form (and thus these structures have only trivial definable functions):

Lemma 3.1 *Let $f : A \subseteq K^k \to K^l$ be an $\mathcal{L}_{dist}$-definable function. There exists a finite partion of A in $\mathcal{L}_{dist}$-cells such that on each cell C the function f has the form*

$$f_{|C} : C \to K^l : x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)),$$

where $f_i(x)$ is either one of the variables $\{x_1, \ldots, x_k\}$ or a constant from K.

Proof. First we note that f can be written as

$$f : A \subseteq K^k \to K^l : x \mapsto (f_1(x), \ldots, f_n(x)),$$

where the coordinate functions $f_i : K^m \to K$ are all $\mathcal{L}_{\text{dist}}$-definable functions. Therefore it is enough to prove the lemma for the case $l = 1$. A function $f : A \subseteq K^k \to K$ is $\mathcal{L}_{\text{dist}}$-definable if and only if

$$\text{Graph} f = \{(x,t) \in A \times K \mid t = f(x)\}$$

is an $\mathcal{L}_{\text{dist}}$-definable set. This means there exists a finite partition of Graph f in $\mathcal{L}_{\text{dist}}$-cells G of the form

$$\{(x,t) \in D \times K \mid \operatorname{ord} a_1(x) \,\square_1\, \operatorname{ord}(t - c(x)) \,\square_2\, \operatorname{ord} a_2(x),\ t - c(x) \in \lambda Q_{n,m}\},$$

with $c(x)$ a constant from K or a variable from $\{x_1, \ldots, x_k\}$. But since f is a function, for each $x \in D$ there must be a unique t such that $(x,t) \in G$. This uniqueness condition implies that $\lambda = 0$, and thus G must have the form

$$G = \{(x,t) \in D \times K \mid t = c(x)\}.$$

$\square$

3.2 Cell decomposition and Skolem functions

When studying definable functions, another natural question to ask is whether a language $\mathcal{L}$ has definable Skolem functions: for a given definable function $f : X \to Y$, does there exist a definable function $g : f(X) \to X$ such that $f \circ g = \text{Id}_{f(X)}$?

In the P-minimal context, Mourgues showed that a P-minimal structure $(K, \mathcal{L})$ has definable Skolem functions if and only if the structure allows cell decomposition, using a notion of cells similar to what we called 'strong cells', i.e. using cells that are defined using only definable functions. When K is p-adically closed, this result can be extended to extensions of $(K, \mathcal{L}_{\text{dist}})$, if scalar multiplication by elements of the following set is definable:

Lemma-Definition 3.2 *The algebraic closure $\overline{\mathbb{Q}}^K$ of $\mathbb{Q}$ in K is the field containing all elements of K that are algebraic over $\mathbb{Q}$.*
The field $\overline{\mathbb{Q}}^K$ has the same residue field as K. Let $\operatorname{ord}_K : K \to \Gamma_K$ be the valuation on K. There exists $\pi \in \overline{\mathbb{Q}}^K$ such that $\operatorname{ord}_K(\pi) = 1$.

Proof. It is easy to see that $\overline{\mathbb{Q}}^K$ has the same residue field as K, since each $x \in \mathbb{F}_K$ is a simple root of $X^{q_K} - X$. The claim follows then by Hensel's Lemma. That $\overline{\mathbb{Q}}^K$ contains an element π with $\operatorname{ord}_K(\pi) = 1$, follows by Lemma 3.5 of [14]. □

Theorem 3.3 *Let K be a p-adically closed field. Suppose that $\mathcal{L} \supseteq \mathcal{L}_{dist}$ and that multiplication by constants from $\overline{\mathbb{Q}}^K$ is definable in $(K, \mathcal{L})$. The structure $(K, \mathcal{L})$ admits strong cell decomposition if and only if $(K, \mathcal{L})$ has definable Skolem functions.*

In fact, we actually obtain strong cell decomposition using continuous functions, since every structure is a reduct of a P-minimal structure.

Proof. First assume that $(K, \mathcal{L})$ has definable Skolem functions. This implies that, if $(K, \mathcal{L})$ has cell decomposition, say using $(\mathcal{L}, \Delta)$-cells, then it admits strong cell decomposition. Indeed, let $f : K^k \to K$ be a function in Δ. Since f is order-definable in $\mathcal{L}$, the following set is $\mathcal{L}$-definable:

$$A := \{(x, t) \in K^{k+1} \mid \operatorname{ord} f(x) = t\}.$$

Consider the projection map $\pi_x : A \to K^k : (x, t) \mapsto x$. Since $(\mathcal{L}, K)$ has definable Skolem functions, there exists a definable function $g : \operatorname{Im} \pi_x \to A : x \mapsto (x, a(x))$ such that $(\pi_x \circ f)(x) = x$ for all $x \in \operatorname{Im} \pi_x$. But then $a(x)$ is a definable function, such that for each $x \in K^k$, $\operatorname{ord} f(x) = \operatorname{ord} a(x)$.

We can now show that $(K, \mathcal{L})$ has strong cell decomposition, using essentially the same proof as that of Mourgues. For this reason, we will only give a brief sketch of the proof, and refer the reader to [11] for details. Let $\mathcal{L}$ be an extension of $\mathcal{L}_{\text{dist}}$, and $S' \subseteq K^{n+1}$ an $\mathcal{L}$-definable set, defined by a formula $\phi(y, x)$. As in [11], using a compactness argument, it can be shown that there exists a quanifierfree $\mathcal{L}_{\text{dist}}$-formula $\psi(z, x)$ such that

$$K \models \forall y \exists z \forall x (\phi(y, x) \Leftrightarrow \psi(z, x)). \tag{5}$$

Write $\pi_n : K^{n+1} \to K^n$ for the projection onto the first n coordinates. As in Lemma 3.3 of [11], it can follows from (5) that, if $(K, \mathcal{L})$ has definable skolem functions, there exists m, an $\mathcal{L}_{\text{dist}}$-definable subset S of K^{m+1} and an $\mathcal{L}$-definable function $f : \pi_n(S') \to K^m$ such that for any $y \in \pi_n(S')$,

$$\{x \in K \mid (y, x) \in S'\} = \{x \in K \mid (f(y), x) \in S\}.$$

Now use the same reasoning as in the proof of Theorem 3.5 of [11], reducing to $\mathcal{L}_{\text{dist}}$-cell decomposition instead of semi-algebraic cell decomposition. The only thing that needs some care is to check that the cell decomposition obtained is a strong decomposition, but this is an immediate consequence of the observation at the start of this proof.

Next, assume that every function in Δ is $\mathcal{L}$-definable. If suffices to check that given an $(\mathcal{L}, \Delta)$-cell C and the projection map $\pi_x : C \subset K^{l+1} \to K^l$, there exists a definable function $g : \pi_x(C) \to C$ such that $\pi_x \circ g = \text{Id}_{\pi_x(C)}$.
If the cell C has a center $c(x) \neq 0$, we first apply a translation

$$C \to C' : (x, t) \mapsto (x, t - c(x)),$$

to a cell C' with center $c'(x) = 0$. Since this translation is bijective, it is invertible. Therefore the problem is reduced to the following. Let C be a cell of the form

$$C = \{(x, t) \in D \times K \mid \operatorname{ord} b(x) \,\square_1 \operatorname{ord} t \,\square_2 \operatorname{ord} a(x),\ t \in \lambda Q_{n,m}\},$$

where $a(x), b(x)$ are $\mathcal{L}$-definable functions and D is an $(\mathcal{L}, \Delta)$-precell. We must show that there exists a definable function $g : \pi_x(C) \to C$ such that $\pi_x \circ g = \text{Id}_{\pi_x(C)}$.

Given $x \in \pi_x(C) \subseteq D$, we have to find $t(x)$ such that $(x, t(x))$ satisfies the conditions

$$\operatorname{ord} b(x) \,\square_1 \operatorname{ord} t(x) \,\square_2 \operatorname{ord} a(x) \tag{6}$$

$$t(x) \in \lambda Q_{n,m} \tag{7}$$

If $\lambda = 0$, put $g(x) = (x, 0)$. From now on we assume that $\lambda \neq 0$.
If $\square_1 = \square_2 =$ 'no condition', we can simply put $g(x) = (x, \lambda)$.
If $\square_2 = <$, we can define g as follows. First partition $\pi_x(C)$ in parts D_μ, such that

$$D_\mu = \{x \in \pi_x(C) \mid a(x) \in \mu Q_{n,m}\}.$$

(Note: if $\mu = 0$, we can reduce to the cases were $\square_2 =$ 'no condition'.) Our strategy is based on the fact that for every $x \in D$, there exists $k \in \mathbb{Z}$ such that k satisfies

$$\operatorname{ord} b_1(x) \,\square_1 \operatorname{ord} \lambda + kn < \operatorname{ord} a(x).$$

Restricting to a set D_μ, we construct an element $t(x)$ with order as close as possible to $\operatorname{ord} a(x)$. This ensures that $t(x)$ satisfies (6). The definiton of g on D_μ will depend on the respective orders of λ and μ.

- If $\operatorname{ord}\lambda < \operatorname{ord}\mu$, we can define $g_{|D_\mu}$ as $g_{|D_\mu} : D_\mu \to C : x \mapsto \left(x, \frac{\lambda}{\mu}a(x)\right)$. This means that we put $t(x) = \frac{\lambda}{\mu}a(x)$. Clearly $t(x) \in \lambda Q_{n,m}$. Also, since $-n < \operatorname{ord}(\frac{\lambda}{\mu}) < 0$, we have that $0 < \operatorname{ord}\frac{a(x)}{t(x)} < n$, and thus condition (6) must be satisfied.

- If $\operatorname{ord}\lambda \geqslant \operatorname{ord}\mu$, put $g_{D_\mu} : D_\mu \to C : x \mapsto \left(x, \frac{\lambda}{\pi^n\mu}a(x)\right)$.

If $\square_1 = <$ and $\square_2 =$ 'no condition', we choose $t(x)$ with order as close as possible to $\operatorname{ord} b(x)$. More specifically, if $\operatorname{ord}\lambda \leqslant \mu$, define g as $g_{D_\mu} : D_\mu \to C : x \mapsto \left(x, \frac{\lambda\pi^n}{\mu}b(x)\right)$, and if $\operatorname{ord}\lambda > \operatorname{ord}\mu$, put $g_{D_\mu} : D_\mu \to C : x \mapsto \left(x, \frac{\lambda}{\mu}b(x)\right)$. $\square$

If we omit the condition that K has to be p-adically closed, we obtain a weaker version of this theorem: we can no longer be assured that definable skolem functions imply the existence of cell decomposition, but if such a decomposition exists, it will be a strong decomposition.

As before, Skolem functions will only be definable if we can define multiplication by enough scalars. Write $\mathbb{P}_K$ for the prime field of K. Choose a generator π_K of R_K. If the residue field $\mathbb{F}_K = \mathbb{F}_p[\overline{a_1}, \ldots, \overline{a_d}]$, choose elements $a_i \in R_K$ such that $\operatorname{ord}(a_i - \overline{a_i}) > 0$. Then put $\mathbb{K}_K := \mathbb{P}_K[\pi_k, a_1, \ldots, a_d]$.

Corollary 3.4 *Suppose that $\mathcal{L} \supseteq \mathcal{L}_{dist}$ and that multiplication by constants from $\mathbb{K}_K$ is definable in $(K, \mathcal{L})$.*
Let Δ be a collection of order-definable functions such that the structure $(K, \mathcal{L})$ has cell decomposition using $(\mathcal{L}, \Delta)$-cells. The following statements are then equivalent:

1. *There exists a collection of $\mathcal{L}$-definable functions Δ' such that $(K, \mathcal{L})$ has cell decomposition using $(\mathcal{L}, \Delta')$-cells,*

2. *The structure $(K, \mathcal{L})$ has definable Skolem functions.*

The condition that multiplication by constants from $\mathbb{K}_K$ be definable in $(K, \mathcal{L})$, is needed: for example the structure $(K; +, -, D^{(4)}, \{R_{n,m}\}_{n,m})$ does not have definable Skolem functions for most $(\mathbb{F}_q, \mathbb{Z})$-fields because we cannot define multiplication by enough scalars (in this structure, scalar multiplication is only definable for elements of $\mathbb{P}_K$).

4 Subaffine structures

In this section we study some expansions of the language $\mathcal{L}_{\text{dist}}$ (or rather $\mathcal{L}'_{\text{dist}}$ as we would like to achieve quantifier elimination whenever possible). We call these expansions sub**affine** because we will only be considering structures $(K, \mathcal{L})$ that are affine in the sense that there does not exist any open subset of K^2 on which multiplication is $\mathcal{L}$-definable. They are **sub**affine because addition should not be definable on all of K^2.

A first, rather trivial example of such an expansion is the language we obtain by adding symbols $\overline{c}$ for the scalar multiplication $\overline{c} : x \mapsto cx$.

$$\mathcal{L}_K := \{\overline{c}\}_{c \in K} \cup \mathcal{L}'_{\text{dist}}.$$

Take fields F, K and $q, q' \in \mathbb{N}$ such that F is an $(\mathbb{F}_q, \mathbb{Z})$-field, K is an $(\mathbb{F}_{q'}, \mathbb{Z})$-field and $F \supset K$. Define the set $\Delta_{K,F}$ to be $\bigcup_{k \in \mathbb{N}} \Delta^k_{K,F}$, where $\Delta^k_{K,F}$ is the following set of polynomials

$$\Delta^k_{K,F} := \{ax + by \mid a, b \in K; x, y \in \{x_1, \ldots, x_k\} \cup F\}.$$

It is easy to see that the structure $(F, \mathcal{L}_K)$ has cell decomposition and quantifier elimination using $(\mathcal{L}_K, \Delta_{K,F})$-cells. The proof is almost literally the same as the corresponding proof for $\mathcal{L}_{\text{dist}}$. Moreover, the scalar multiplication functions we added are in fact the only non-trivial functions, or to be more precise:

Lemma 4.1 *Let $f : A \subseteq F^k \to F^l$ be an $\mathcal{L}_K$-definable function. There exists a finite partition of A in $(\mathcal{L}_K, \Delta_{K,L})$-cells such that on each cell C the function f has the form*

$$f_{|c} : C \to K^l : x \mapsto (f_1(x), f_2(x), \ldots, f_l(x)),$$

where $f_i(x)$ is either a constant from F or $f_i(x) = ax_j$, with $a \in K$ and x_j one of the variables $\{x_1, \ldots, x_k\}$.

Structures $(F, \mathcal{L}_K)$ do not have definable Skolem functions, as can be seen from the following counterexample:

Lemma 4.2 *Let Π be the projection map*

$$\Pi : A := \{(x, y, z) \in F^3 \mid \operatorname{ord} z = \operatorname{ord}(y - x)\} \to F^2 : (x, y, z) \mapsto (x, y).$$

There exists no $\mathcal{L}_K$-definable function f such that $\Pi \circ f = \mathrm{Id}_{\mathrm{Im}_\Pi}$.

Proof. Suppose that such a function f exists. Up to a finite partition of $\pi(A)$ in cells $\bigcup C_i \cup \bigcup D_j$, this function will be of one of the forms

$$f_{|C_i} : C \to F^3 : (x, y) \mapsto (x, y, a_i) \text{ or } f_{|D_j} : D_j \to F^3 : (x, y) \mapsto (x, y, b_j x_j),$$

with $a_i \in F$, $b_j \in K$ and x_j is one of the variables x and y.
On cells C_i, we use a function of the form $(x, y) \mapsto (x, y, a_i)$, which implies that $\operatorname{ord}(x - y) = \operatorname{ord} a_i$ for all $(x, y) \in C_i$. As our partition is finite, we can only have a finite number of cells of this type. Put $M := \max_i \operatorname{ord} a_i$, then all tuples (x, y) for which $\operatorname{ord}(x - y) > M$ will be contained in $\bigcup D_j$. For each $k > 0$, this set contains elements (x, y) that satisfy

$$\operatorname{ord} x = \operatorname{ord} y < M \wedge \operatorname{ord} x - y = \operatorname{ord} x + k,$$

which means that we would need a partition in an infinite number of parts D_j to define f. $\square$

This counterexample suggests that it might be impossible to have definable Skolem functions in a language where addition is not definable. This is our main motivation for studying subaffine structures: we will consider a language that has a restricted form of addition, and see whether this languages allows us to define Skolem functions. We will consider the following the functions $\boxplus_R$ and $\boxminus_R$, defined by

$$\boxplus_R : K^2 \to K : (x,y) \mapsto \begin{cases} x+y & x,y \in R_K \\ 0 & \text{otherwise,} \end{cases}$$

and analogously for $\boxminus_R$, with $+$ replaced by $-$.

For these functions, we will study the language

$$\mathcal{L}_{\boxplus,K} := \{\boxplus, \boxminus\} \cup \mathcal{L}_K.$$

Let $F \supset K$ be $(\mathbb{F}_q, \mathbb{Z})$-fields, resp. $(\mathbb{F}_{q'}, \mathbb{Z})$-fields. We will verify that a structure $(F, \mathcal{L}_{\boxplus_R,K})$ has cell decomposition and quantifier elimination for the language

$$\mathcal{L}_{\boxplus_R,K} := \{\boxplus_R, \boxminus_R\} \cup \mathcal{L}_K.$$

Definition 4.3 *Let $(F, \mathcal{L}_{\boxplus_R,K})$ be a structure.*
Write $\mathrm{Poly}_{\boxplus_R,K}$ *for the set of functions that can be defined as a composition of the functions $\boxplus_R, \boxminus_R$ and $\overline{c}$ for $c \in K$, combined with variables $x_1, x_2, \ldots$ and constants from F.*

It is important to stress that these expressions do not entirely behave like polynomials. More precisely, we have to be aware that distributivity does not always hold. For example: suppose $0 < k_1 < k_2 < k_3$, then

$$\pi^{-k_2}(\pi^{k_1} \boxplus_R \pi^{k_3}) \neq \pi^{k_1-k_2} \boxplus_R \pi^{k_3-k_2} = 0.$$

First we need to define a notion of cells for this context.

Definition 4.4 *Let $\Delta_{\boxplus_R,K}$ be the set*

$$\bigcup_{r\in\mathbb{N}} \{a(x_1,\ldots,x_r) - b(x_1,\ldots,x_r) \mid a(x), b(x) \in \mathrm{Poly}_{\boxplus_R,K}\}.$$

A subset of F^k is called a $(\boxplus_R, K)$-cell if it is a $(\mathcal{L}_{\boxplus_R,K}, \Delta_{\boxplus_R,K})$-cell and the center is a function from $\mathrm{Poly}_{\boxplus_R,K}$.

In the next lemmas, we will show that, up to a finite partition in cells, $\mathrm{Poly}_{\boxplus_R,K}$-functions can always be written in a fairly simple way. Note that for every $\gamma_0 \in \Gamma_K$, the following function is (quantifier free) definable:

$$\boxplus_{\gamma_0} : (x,y) \mapsto \begin{cases} x+y & \mathrm{ord}\, x, \mathrm{ord}\, y \geqslant \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we have the following calculation rule. For every $a \in K; b,c \in F, \gamma \in \Gamma_K$:

$$a(b \boxplus_\gamma c) = ab \boxplus_{\gamma + \mathrm{ord}\, a} ac.$$

Lemma 4.5 *Take $\gamma \in \Gamma_K, a \in K$; $d(x), h(x) \in \mathrm{Poly}_{\boxplus_R,K}$. Let $\Box$ denote '$<$', '$\leqslant$', '$>$' or '$\geqslant$'. The set*

$$S := \{(x,t) \in F^{k+1} \mid \mathrm{ord}\,(at \boxplus_\gamma d(x)) \;\Box\; \mathrm{ord}\, h(x)\}$$

can be partitioned as a finite union of $(\boxplus_R, K)$-cells.

Proof. First note that we may suppose that $a = 1$ (if $at \neq 0$), since

$$\mathrm{ord}\,(at \boxplus_\gamma d(x)) \;\Box\; \mathrm{ord}\, h(x) \Leftrightarrow \mathrm{ord}\left(t \boxplus_{\gamma - \mathrm{ord}\, a} \frac{d(x)}{a}\right) \Box\; \mathrm{ord}\, \frac{h(x)}{a}.$$

Put $a = 1$. The set S can then be partitioned as the union of the following three sets:

$$\begin{aligned} S \;\; = \;\; & \{(x,t) \in F^{k+1} \mid \mathrm{ord}\, t < \gamma \wedge \mathrm{ord}\, 0 \,\Box\, \mathrm{ord}\, h(x)\} \\ & \cup \{(x,t) \in F^{k+1} \mid \mathrm{ord}\, t \geqslant \gamma \wedge \mathrm{ord}\, d(x) < \mathrm{ord}\,\gamma \wedge \mathrm{ord}\, 0 \,\Box\, \mathrm{ord}\, h(x)\} \\ & \cup \big(\{(x,t) \in F^{k+1} \mid \mathrm{ord}\, t \geqslant \gamma \wedge \mathrm{ord}\, d(x) \geqslant \gamma\} \\ & \qquad \cap \{(x,t) \in F^{k+1} \mid \mathrm{ord}\,(t + d(x)) \;\Box\; \mathrm{ord}\, h(x)\}\big) \end{aligned}$$

The first two sets are cells. The third set is the intersection of two $(\boxplus_R, K)$-cells and thus again a finite union of cells by Theorem 2.1. $\Box$

Lemma 4.6 *For $x = (x_1, \ldots, x_n)$, and t one variable, let the functions $f_1(x,t)$, $\ldots$, $f_r(x,t)$ be in $\mathrm{Poly}_{\boxplus_R,K}$. F^{k+1} can be partitioned in a finite number of cells A, such that on each cell A there are $\gamma_1, \ldots, \gamma_r \in \Gamma_K$, such that either*

$$f_i(x,t) = h_i(x) \quad \text{or} \quad f_i(x,t) = a_i t \quad \text{or} \quad f_i(x,t) = a_i t \boxplus_{\gamma_i} d_i(x),$$

with $a_i \in K$, and $h_i(x), d_i(x)$ are in $\mathrm{Poly}_{\boxplus_R,K}$.

.

Proof. We will work by induction on the number of compositions. Suppose the lemma holds for functions f and g. It suffices to check that the lemma also holds for $\overline{c} \circ f$ and $f \boxplus_R g$. Take a suitable decomposition into cells A. Choose $c \in K$. For all $(x,t) \in A$, the function $\overline{c} \circ f$ will have one of the following forms: either

$$(\overline{c} \circ f)(x,t) = \overline{c} \circ h_f(x) \quad \text{or} \quad (\overline{c} \circ f)(x,t) = c a_f t, \tag{8}$$

or

$$(\overline{c} \circ f)(x,t) = c(a_f t \boxplus_{\gamma_f} d_f(x)). \tag{9}$$

If we have functions as in (8), we are done. We can rewrite (9) as

$$c(a_f t \boxplus_{\gamma_f} d_f(x)) = c a_f t \boxplus_{\gamma_f + \mathrm{ord}\, c} (\overline{c} \circ d_f(x)).$$

We can apply a similar reasoning to the function $(f \boxplus_R g)$. In most cases, it is obvious that the function has one of the required forms. The only nontrivial cases are when $(f \boxplus_R g)$ has one of the following forms for $(x,t) \in A$:

$$(f \boxplus_R g)(x,t) = \begin{cases} a_f t \boxplus_R (a_g t \boxplus_{\gamma_g} d_g(x)) & \text{(case 1)} \\ h_f(x) \boxplus_R (a_g t \boxplus_{\gamma_g} d_g(x)) & \text{(case 2)} \\ (a_f t \boxplus_{\gamma_f} d_f(x)) \boxplus_R (a_g t \boxplus_{\gamma_g} d_g(x)) & \text{(case 3)} \end{cases}$$

Remember that the set $\{(x,t) \in K^{k+1} \mid \operatorname{ord}(at \boxplus_\gamma d(x)) \geqslant 0\}$ can be written as a finite union of cells, by Lemma 4.5.

We will check that the lemma holds in case 3 (Case 1 and 2 are similar). Partition A further in cells such that either $\operatorname{ord}(a_f t \boxplus_{\gamma_f} d_f(x)) < 0$, or $\operatorname{ord}(a_f t \boxplus_{\gamma_f} d_f(x)) \geqslant 0$ for all $(x,t) \in A$ (and similarly for g). We only need to consider cells where $\operatorname{ord}(a_f t \boxplus_{\gamma_f} d_f(x)) \geqslant 0$ and $\operatorname{ord}(a_g t \boxplus_{\gamma_g} d_f(x)) \geqslant 0$ as our claim is trivially true on other cells. Partition these cells further depending on the order of $a_f t, a_g t, d_f(x), d_g(x)$. The only case that is not immediately obvious is when

$$\operatorname{ord} a_f t \geqslant \gamma_f, \operatorname{ord} d_f(x) \geqslant \gamma_f, \operatorname{ord} a_g t \geqslant \gamma_g \text{ and } \operatorname{ord} d_g(x) \geqslant \gamma_g.$$

Let C be such a cell. Without loss of generality, we may suppose that $\gamma_f \leqslant \gamma_g$. For $(x,t) \in C$ we find that

$$\begin{aligned} (f \boxplus_R g)_{|C} &= (a_f t \boxplus_{\gamma_f} d_f(x)) \boxplus_R (a_g t \boxplus_{\gamma_g} d_g(x)) \\ &= (a_f + a_g)t + (d_f(x) + d_g(x)) \\ &= (a_f + a_g)t \boxplus_{\gamma_f} (d_f(x) \boxplus_{\gamma_f} d_g(x)). \end{aligned}$$

□

Proposition 4.7 *Any $\mathcal{L}_{\boxplus_R}$-definable set can be partitioned as a finite union of $(\boxplus_R, K)$-cells.*

Proof. First we show that quantifier-free definable sets can be partitioned as a finite union of cells. Because of Theorem 2.1, it is sufficient to check that sets of type S_1 or S_2 can be partitioned as a finite union of cells:

$$\begin{aligned} S_1 &:= \{(x,t) \in F^{k+1} \mid f_1(x,t) - f_2(x,t) \in \lambda Q_{n,m}\}, \\ S_2 &:= \{(x,t) \in F^{k+1} \mid \operatorname{ord} f_1(x,t) - f_2(x,t) < \operatorname{ord} f_3(x,t) - f_4(x,t)\}, \end{aligned}$$

where the $f_i(x,t)$ are functions from $\mathrm{Poly}_{\boxplus_R, K}$. Intersect the sets S_1 and S_2 with sets $\{(x,t) \in F^{k+1} \mid \operatorname{ord} f_i(x,t) \,\square\, \operatorname{ord} f_j(x,t)\}$ or $\{(x,t) \in F^{k+1} \mid \operatorname{ord} f_i(x,t) \,\square\, \operatorname{ord} h_j(x)\}$, where $\square$ may denote $<, =$ or $>$. When we apply Theorem 2.1 and Proposition 4.6 to these intersections, it is easy to see that it suffices to check that the sets $\widetilde{S_1}$ and $\widetilde{S_2}$

$$\begin{aligned} \widetilde{S_1} &:= \{(x,t) \in F^{k+1} \mid a_1 t \boxplus_{\gamma_1} d_1(x) \in \lambda Q_{n,m}\}, \\ \widetilde{S_2} &:= \{(x,t) \in F^{k+1} \mid \operatorname{ord}(a_2 t \boxplus_{\gamma_2} d_2(x)) < \operatorname{ord}(a_3 t \boxplus_{\gamma_3} d_3(x))\}, \end{aligned}$$

can be partitioned as a finite union of cells for all $\gamma_i \in \Gamma_K$ and $d_i(x) \in \mathrm{Poly}_{\boxplus_R,K}$. For $\widetilde{S_1}$ this follows from the observation that the expression $at \boxplus_\gamma d(x) \in \lambda Q_{n,m}$ is equivalent with

$$\begin{aligned}&[\ \mathrm{ord}\, at < \gamma \wedge \lambda = 0] \vee [\mathrm{ord}\, at \geqslant \gamma \wedge \mathrm{ord}\, d(x) < \gamma \wedge \lambda = 0]\\ \vee &\left[\ \mathrm{ord}\, at \geqslant \gamma \wedge \mathrm{ord}\, d(x) \geqslant \gamma \wedge \left(t - \frac{-d(x)}{a}\right) \in \frac{\lambda}{a} Q_{n,m}\right].\end{aligned}$$

For the set $\widetilde{S_2}$, note that we can restrict our attention to $\widetilde{S_2}^{(1)} := \widetilde{S_2} \cap A_\geqslant$, with

$$A_\geqslant := \{(x,t) \in F^{k+1}\ |\mathrm{ord}\, a_i t \geqslant \gamma_i \wedge \mathrm{ord}\, d_i(x) \geqslant \gamma_i,\ \text{for } i \in \{2,3\}\},$$

since it follows easily from Theorem 2.1 that $\widetilde{S_2} \backslash \widetilde{S_2}^{(1)}$ can be partitioned as a finite union of cells. Write $d_i'(x) = \frac{-d_i(x)}{a_i}$. Now $\widetilde{S_2}^{(1)}$ is equal to the set

$$\widetilde{S_2}^{(1)} = \{(x,t) \in A_\geqslant \mid \mathrm{ord}\, a_2(t - d_2'(x)) < \mathrm{ord}\, a_3(t - d_3'(x))\}.$$

For $\square$ = '$<$', '$=$', or '$>$', put

$$B_\square := \{(x,t) \in K^{k+1} \mid \mathrm{ord}\,(t - d_2'(x))\ \square\ \mathrm{ord}\,(d_2'(x) - d_3'(x)\}.$$

Then $\widetilde{S_2}^{(1)} = \left(\widetilde{S_2}^{(1)} \cap B_<\right) \cup \left(\widetilde{S_2}^{(1)} \cap B_=\right) \cap \left(\widetilde{S_2}^{(1)} \cap B_>\right)$.
Now if $\mathrm{ord}\,(t - d_2'(x)) < \mathrm{ord}\,(d_2'(x) - d_3'(x))$, then $\mathrm{ord}\,(t - d_3'(x)) = \mathrm{ord}\,(t - d_2'(x))$, so

$$\widetilde{S_2}^{(1)} \cap B_< = A_\geqslant \cap B_< \cap \{(x,t) \in F^{k+1}\ |\mathrm{ord}\, a_2 < \mathrm{ord}\, a_3\}.$$

By Theorem 2.1, this can be written as a finite union of cells. The situation is similar when we intersect with $B_=$ or $B_>$.

The fact that quantifier-free definable sets can be partitioned as a finite union of cells, also implies that for all $a_i \in \Delta_{\boxplus_R,K}$, the relation $\mathrm{ord}\, a_i \equiv l \mod n$ can be defined using precell relations. Because of this, structures $(F, \mathcal{L}_{\boxplus_R,K})$ have quantifier elimination by Lemma 2.2.

$\square$

The following classification of the definable functions is an immediate consequence of this proposition.

Corollary 4.8 *Let $f : A \subseteq F^\ell \to F^r$ be an $\mathcal{L}_{\boxplus_R,K}$-definable function. There exists a finite partition of A in cells C, such that on each cell C,*

$$f_{|C}(x) = (f_1(x), \ldots f_r(x)),$$

where $f_i(x) \in \mathrm{Poly}_{\boxplus_R,K}$, for $i = 1, \ldots, r$.

Proposition 4.9 *The addition function $+ : F^2 \to F : (x,y) \mapsto x + y$ is not definable in $\mathcal{L}_{\boxplus_R,K}$.*

Proof. Suppose addition is definable, say by some function f. Applying Corollary 4.8 and Lemma 4.6, We can partition F^2 in cells C_i and D_i such that

$$f_{|C_i}(x, y) = a_i x \boxplus_{\gamma_{i,1}} (b_i y \boxplus_{\gamma_{i,2}} c_i),$$

and

$$f_{|D_i}(x, y) = a_i x \quad \text{or} \quad f_{|D_i}(x, y) = b_i y \quad \text{or} \quad f_{|D_i}(x, y) = c_i,$$

for some $a_i, b_i, \in K, c_i \in F$ and $\gamma_{i,1}, \gamma_{i,2} \in \Gamma_K$. The precise value of these constants depends on C_i. Put $\gamma := \min_i\{\gamma_{i,1}\}$. Clearly all elements (x, y) for which $\operatorname{ord} a_i x < \gamma$ must be contained in the cells D_i since for such elements,

$$f_{|C_i}(x, y) = 0 \neq x + y.$$

It is clear that the functions $f_{|D_i}$ cannot be used to define addition on a large enough set, which proves that the addition function '+' is not definable. □

The fact that addition is not definable is caused by the fact that we have restricted multiplication to multiplication by a constant. More precisely, the reason is the following (for simplicity, suppose that Γ_K is $\mathbb{Z}$). In our language, it is impossible to take a 'limit' for $\operatorname{ord} x$ going to $-\infty$. In a language with normal multiplication, we do not have this restriction, and as a consequence '+' can easily be defined in such a language. Take for example the language $\mathcal{L} = (\boxplus_R, \cdot)$. For any $x, y \in K$ with $\operatorname{ord} x \leqslant \operatorname{ord} y$ and $x \neq 0$, we can define addition using the following equality.

$$x + y = x\left(1 \boxplus_R \frac{y}{x}\right).$$

It is also not hard to see that definable Skolem functions do not always exist for $\mathcal{L}_{\boxplus_R,K}$. Indeed, this follows from the following counterexample.

Lemma 4.10 *Let* Π *be the projection map*

$$\Pi : \{(x, y, z) \in F^3 \mid \operatorname{ord} z = \operatorname{ord}(y - x)\} \to K^2 : (x, y, z) \mapsto (x, y)\}.$$

There exists no $\mathcal{L}_{\boxplus_R,K}$*-definable function* f *such that* $\Pi \circ f = \mathrm{Id}_{\mathrm{Im}_\Pi}$.

Proof. Put $A = \{(x, y, z) \in F^3 \mid \operatorname{ord} z = \operatorname{ord}(y - x)\}$. Suppose $f : \Pi(A) \to A$ is a definable function for which $\Pi \circ f = \mathrm{Id}_{\mathrm{Im}_\Pi}$. By Corollary 4.8, there exists a partition of $\Pi(A)$ in cells C_i and D_i such that

$$f_{|C_i}(x, y) = (x, y, a_i x \boxplus_{\gamma_i} h_i(y)),$$

and

$$f_{|D_i}(x, y) = (x, y, a_i x), \quad \text{or} \quad f_{|D_i}(x, y) = (x, y, h_i(y)),$$

with $a_i \in K, \gamma_i \in \Gamma_K$ and $h_i \in \mathrm{Poly}_{\boxplus_R,K}$. Note that $f_{|C_i}(x, y) = (x, y, 0)$ for elements (x, y) for which $\operatorname{ord} a_i x < \gamma_i$. So the sets D_i must contain all (x, y) for which $\operatorname{ord} x$ is too small. However, it is easy to see that the functions $f_{|D_i}$ do

not satisfy our requirements. $\square$

This is a confirmation of our conjecture that a structure where addition is not definable does not have definable Skolem functions. Take for example the structure $(F, \mathcal{L}_{\boxplus_R, F})$. If we fix any constant $\gamma \in \Gamma_F$, we can define addition for the set $\{(x, y) \in F^2 \mid \min\{\operatorname{ord} x, \operatorname{ord} y\} \geqslant \gamma\}$. Taking smaller and smaller values for γ, we can thus define addition on very large open subsets of F^2, but still not large enough to enable us to define Skolem functions.

Acknowledgements

The results presented in this paper were obtained as part of my PhD thesis. I would like to thank my supervisor, Raf Cluckers, for many stimulating conversations about this topic, and other members of the jury (in particular, Jan Denef, Angus Macintyre and Leonard Lipshitz) for useful comments. Many thanks also to the Math Department of K.U.Leuven, for providing financial support and a very stimulating working environment.

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